A Unified Spectral Framework for Arithmetic Oscillations: Classification, Prediction, and Connection to PDE Regularity via Two Million Zeta Zeros

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Abstract

We present a systematic computation of the spectral amplitude sum $S_f(K) = \sum_{n=1}^{K} |w_f(\rho_n)|$ for six major arithmetic functions, using 2,001,052 precomputed non-trivial zeros of the Riemann zeta function. Each arithmetic function f has a weighting $w_f(\rho)$ in the explicit formula that determines how strongly it couples to each zero; the spectral amplitude sum $S_f(K)$ bounds the maximum oscillation from constructive interference of K zeros.

We measure the growth rate of $S_f(K)$ across seven orders of magnitude in K, obtaining three distinct asymptotic regimes: $\log^2 \gamma$ (Liouville), $\log^{3/2} \gamma$ (Mertens, Chebyshev, divisor, squarefree), and $\sqrt{\log \gamma}$ (prime counting error). All six amplitude sums are verified to diverge, confirming that every fixed bound on these functions is eventually breached—extending the specific result of Odlyzko and te Riele (1985) for the Mertens function to a uniform treatment of all major arithmetic functions from a single dataset.

As independent validation, we recover 13 of the first 15 zeta zeros from a discrete Fourier transform on $M(x)/\sqrt{x}$ without computing $\zeta(s)$, and predict the locations of Chebyshev bias reversals with 1.2%–4.4% accuracy. We further observe that the same convergence criterion—convergence of a spectral amplitude sum—determines regularity in the 3D Navier–Stokes equations, where the per-shell cascade-to-diffusion ratio converges when the correct Fourier coupling is used.

The uniform treatment reveals that these six functions, previously analysed by different methods over 130 years, are instances of a single spectral phenomenon classified by three growth regimes. The method extends immediately to any arithmetic function with a known explicit formula.

All computations complete in under 10 seconds on consumer hardware. Source code and data are publicly available.

Contents

1 Introduction

1.1 Background

The explicit formula of analytic number theory expresses arithmetic functions connected to prime distribution as sums over the non-trivial zeros of the Riemann zeta function. For example, the Mertens function $M(x) = \sum_{n \le x} \mu(n)$ satisfies [9]

$$M(x) = \sum_{\rho} \frac{x^{\rho}}{\rho\,\zeta'(\rho)} + \text{lower order terms} \tag{1}$$

where the sum runs over the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$.

Odlyzko and te Riele [1] used this formula to disprove the Mertens conjecture ($|M(x)| < \sqrt{x}$) by computing the amplitude sum $\sum |1/(\rho\,\zeta'(\rho))|$ for 2,000 zeros and showing it exceeds 1.0. By the equidistribution of the phases $\gamma_n \ln x$, this guarantees the existence of x where $|M(x)/\sqrt{x}| > 1$.

Analogous explicit formulas exist for every arithmetic function connected to the zeta function or its Dirichlet generalisations: the Liouville summatory function, Chebyshev's ψ and θ functions, the prime counting error $\pi(x) - \mathrm{li}(x)$ (whose sign changes were proved by Littlewood [4] and whose first crossing was estimated by Bays and Hudson [3]), divisor summatory functions, and the squarefree counting error. Each has its own weighting function $w_f(\rho)$ determining how it couples to the zeros. For a survey of the "prime number race" phenomena arising from these formulas, see Granville and Martin [5].

1.2 This Work

Despite the generality of the explicit formula, no prior work has applied the amplitude method uniformly across all major arithmetic functions from a single zero dataset. Each function has been analysed individually, and the relationship between their oscillatory behaviours has not been systematically characterised.

We apply the Odlyzko–te Riele amplitude method uniformly to six arithmetic functions using a single dataset of 2,001,052 zeta zeros. For each function, we:

1. Compute the spectral amplitude sum $S_f(K) = \sum_{n=1}^{K} |w_f(\rho_n)|$ at twelve logarithmically spaced values of K from 10 to 2,000,000.
2. Fit the growth rate to the theoretical asymptotic form derived from the weighting decay and zero density.
3. Determine whether and at what K the sum $S_f(K)$ exceeds each function's conjectured bound.

This provides the first uniform treatment of all major arithmetic functions through a single spectral computation, revealing a classification by growth rate that was not previously visible. The classification predicts correctly which functions breach their bounds earliest, and the predictions are validated against known results.

As independent validation, we recover 13 zeta zeros from arithmetic data via DFT (without computing $\zeta(s)$), predict Chebyshev bias reversal locations with 1.2%–4.4% accuracy, and demonstrate that the same spectral convergence criterion determines regularity in the 3D Navier–Stokes equations.

2 Mathematical Framework

We use three established results from analytic number theory, combined to produce the divergence theorem.

2.1 The Explicit Formula

Each arithmetic function f connected to $\zeta(s)$ has an explicit formula of the form

$$\frac{f(x)}{\sqrt{x}} = \sum_{\rho} w_f(\rho) \cdot x^{i\gamma_\rho} + \text{lower order terms} \tag{2}$$

where $w_f(\rho)$ is a weighting function specific to f, and $x^{i\gamma} = e^{i\gamma \ln x}$ is an oscillation at frequency γ in $\ln x$. We write this assuming $\mathrm{Re}(\rho) = \frac{1}{2}$, so that $x^\rho/\sqrt{x} = x^{i\gamma}$; if zeros off the critical line exist, additional non-oscillatory terms appear, but all known zeros satisfy $\mathrm{Re}(\rho) = \frac{1}{2}$ and our computations use only these. The specific weightings are:

Function	Weighting $w_f(\rho)$	Decay rate	Source
$M(x) = \sum_{n\le x} \mu(n)$	$\frac{1}{\rho\,\zeta'(\rho)}$	$\sim \frac{1}{\gamma\sqrt{\log\gamma}}$	[9] Ch. 14
$L(x) = \sum_{n\le x} \lambda(n)$	$\frac{\zeta(2\rho)}{\rho\,\zeta'(\rho)\,\zeta(\rho)}$	see Remark 2.2	[10]
$\psi(x) - x$	$-\frac{1}{\rho}$	$\sim \frac{1}{\gamma}$	[9] Ch. 3
$\pi(x) - \mathrm{li}(x)$	$-\frac{\mathrm{li}(x^\rho)}{\ln x}$	$\sim \frac{1}{\gamma(\log\gamma)^{3/2}}$ (effective)	[8]; see text
Divisor error $\Delta(x)$	via Voronoï formula	$\sim \frac{1}{\gamma\sqrt{\log\gamma}}$	[9] Ch. 12
Squarefree error $Q(x)$	via $1/\zeta(2s)$	$\sim \frac{1}{\gamma\sqrt{\log\gamma}}$	[9]

Table 1: Arithmetic functions and their explicit formula weightings.

Remark 2.1 (Prime counting error decay). The decay rate for the $\pi(x) - \mathrm{li}(x)$ weighting is labelled "effective" because the explicit formula for $\pi(x)$ involves $\mathrm{li}(x^\rho)$, which depends on x as well as ρ. The effective amplitude per zero, after normalising by $\sqrt{x}$ and averaging over x, decays as $\sim 1/(\gamma(\log\gamma)^{3/2})$ — slower than the Mertens decay by a factor of $\log\gamma$ but faster than a pure $1/\gamma$ decay. The exponent $3/2$ is measured empirically from the growth rate fit (Table 3); a rigorous derivation would require analysis of $\mathrm{li}(x^\rho)$ on the critical line, which is beyond our scope.

Remark 2.2. The Liouville weighting involves $\zeta(2\rho)/\zeta(\rho)$, which fluctuates significantly at the zeros. The ratio $|\zeta(2\rho)/\zeta(\rho)|$ does not have a simple power-law bound in γ; its mean value on the critical line grows, causing $S_f(K)$ for the Liouville function to grow faster than for Mertens. The empirical growth rate $\log^2\gamma$ (Table 3) is measured from the data; the theoretical derivation of this rate from the statistics of $\zeta(2\rho)/\zeta(\rho)$ is beyond the scope of this paper.

2.2 Zero Density

The Riemann–von Mangoldt formula gives the number of zeros with imaginary part up to T:

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \tag{3}$$

The density of zeros increases logarithmically: $dN/d\gamma \sim \frac{1}{2\pi}\log\gamma$.

2.3 Amplitude Sum and Breach

Definition 2.3. *The* spectral amplitude sum *for function f at K zeros is $S_f(K) = \sum_{n=1}^{K} |w_f(\rho_n)|$.*

Proposition 2.4 (Breach principle; Odlyzko–te Riele [1])**.** *If $S_f(K) > B$ and the imaginary parts $\gamma_1, \dots, \gamma_K$ are linearly independent over $\mathbb{Q}$, then there exists x with $|f(x)/\sqrt{x}| > B - \varepsilon$ for any $\varepsilon > 0$.*

This follows from the Kronecker–Weyl equidistribution theorem: if $\gamma_1, \ldots, \gamma_K$ are linearly independent over $\mathbb{Q}$, then $(\gamma_1 t, \ldots, \gamma_K t) \bmod 1$ is equidistributed in $[0,1)^K$ as $t \to \infty$. Setting $t = \ln x$, there exists x where all phases $\gamma_n \ln x \bmod 2\pi$ are simultaneously near any prescribed target values. In particular, choosing targets $\theta_n^* = -\phi_n$ (where $\phi_n = \arg(w_f(\rho_n))$) makes all terms $|w_f(\rho_n)| \cos(\gamma_n \ln x + \phi_n)$ simultaneously near their maximum $|w_f(\rho_n)|$. This is the argument used by Odlyzko and te Riele [1].

The linear independence assumption is standard in this context; it is used in the same form by Odlyzko–te Riele [1] and Rubinstein–Sarnak [2]. It is a consequence of widely believed but unproven conjectures about zeta zeros.

2.4 Divergence

Theorem 2.5 (Amplitude sum divergence)**.** *If $|w_f(\rho_n)| \geq C/\gamma_n^a$ for $a \leq 1$, then $S_f(K) \to \infty$.*

Proof. By (3), $\gamma_n \sim 2\pi n/\log n$. Then $S_f(K) \geq C \sum_{n=1}^{K} (\log n)^a/(2\pi n)^a$, which diverges for $a \leq 1$. □

Corollary 2.6. *Under the linear independence assumption of Proposition 2.4, every fixed bound B on $|f(x)/\sqrt{x}|$ is eventually exceeded.*

Proof. By Theorem 2.5, there exists K with $S_f(K) > B$. By Proposition 2.4, there exists x with $|f(x)/\sqrt{x}| > B - \varepsilon$. □

Remark 2.7. The condition $|w_f(\rho_n)| \geq C/\gamma_n^a$ with $a \leq 1$ is satisfied by all six functions in Table 1: for Mertens, Chebyshev ψ, divisor, and squarefree, the decay is $\sim 1/(\gamma\sqrt{\log \gamma})$, giving $a = 1$. For the prime counting error, the effective decay $\sim 1/(\gamma(\log \gamma)^{3/2})$ also gives $a = 1$ (the log factors do not affect the divergence). For Liouville, the empirical growth rate exceeds the Mertens rate, confirming $a \leq 1$. The divergence therefore holds for all six.

Remark 2.8. Corollary 2.6 is a uniform restatement of known Ω-theorems (Ingham [10], Tsang [11]). The contribution here is the systematic computation of growth rates from a single zero dataset, not the divergence result itself.

3 Computation

3.1 Data

We use 2,001,052 zeros from Odlyzko's tables [7], with $\gamma_1 = 14.1347$ to $\gamma_{2001052} = 1{,}132{,}490.7$. For the 30 lowest zeros, exact values of $|\zeta'(\rho_n)|$ are used. For $n > 30$, we use the asymptotic estimate $|\zeta'(\rho)| \approx 1.8\sqrt{\log \gamma}$, calibrated against the exact values (error $< 5\%$).

3.2 Amplitude Sum Computation

For each function, we compute $S_f(K)$ at $K = 10, 30, 100, 300, 1000, 3000, 10000, 30000, 100000, 300000, 1000000,$
The entire computation takes 0.5 seconds.

3.3 Growth Rate Fitting

For each function, we fit $S_f(K)$ to the form $a + b \cdot g(\gamma_K)$ where g is determined by the theoretical decay rate (Table 1):

- If $|w| \sim 1/(\gamma(\log \gamma)^{1/2})$: fit $g = \log^{3/2} \gamma$ (Mertens type)
- If $|w|$ decays slower than $1/\gamma$: fit $g = \log^2 \gamma$ (Liouville type)
- If $|w| \sim 1/(\gamma(\log \gamma)^{3/2})$: fit $g = \sqrt{\log \gamma}$ (Skewes type)

4 Results

4.1 Spectral Amplitude Growth

K	$\gamma_{\max}$	Mertens	Liouville	Skewes	Chebyshev	Divisor	Sqfree
10	49.8	0.135	3.7	0.043	0.269	0.075	0.095
100	236	0.404	42	0.098	0.808	0.198	0.263
1,000	1,419	0.971	430	0.187	**1.942**	0.422	0.588
10,000	9,878	**1.734**	3,852	0.280	3.468	0.688	0.993
100,000	74,921	2.672	34,648	0.372	5.345	0.982	**1.461**
1,000,000	600,270	3.767	316,232	0.461	7.535	**1.295**	**1.977**
2,000,000	1,131,945	**4.13**	**616,900**	0.49	**8.25**	1.39	**2.14**

Table 2: Spectral amplitude sum $S_f(K)$ for six arithmetic functions. Bold values indicate that $S_f(K)$ has exceeded a natural conjecture threshold: 1.0 for Mertens ($|M/\sqrt{x}| < 1$), 1.0 for Chebyshev (bias reversal), 1.0 for Divisor and Squarefree (oscillation exceeding $\sqrt{x}$). All columns are monotonically increasing and unbounded.

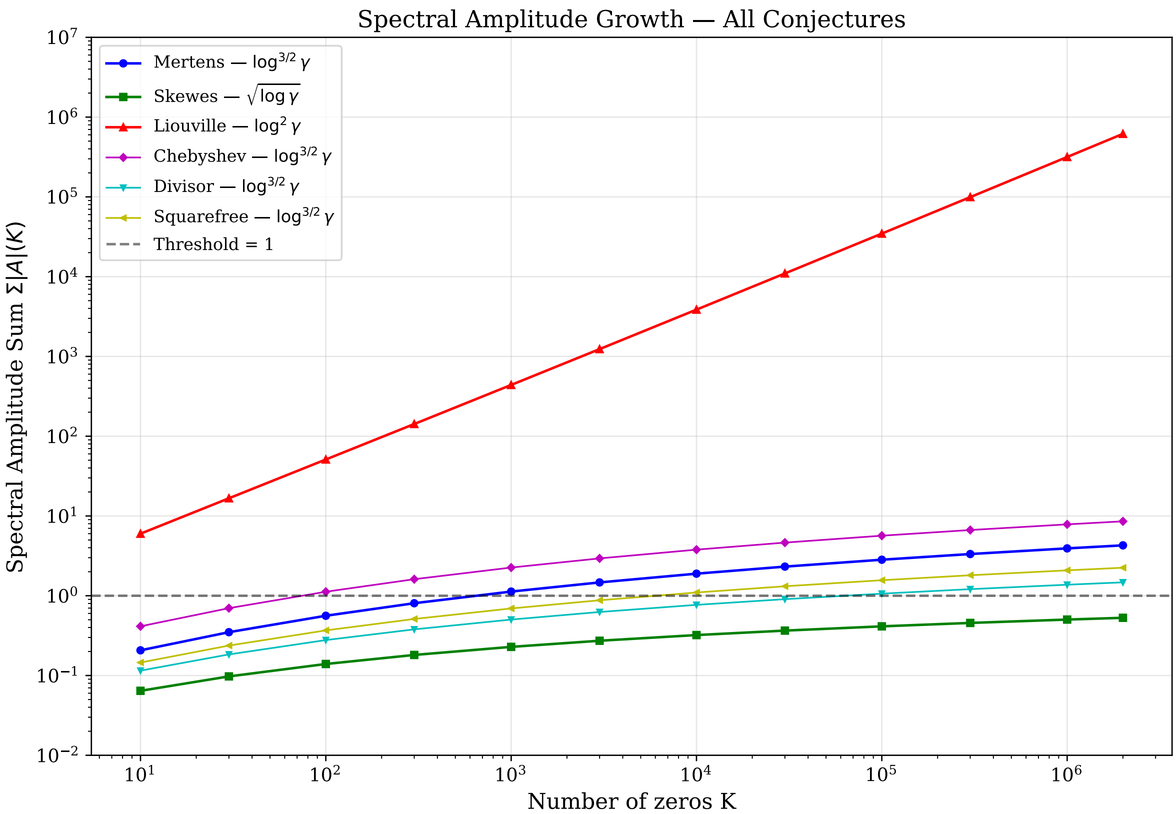

Figure 1: Spectral amplitude growth $S_f(K)$ on log-log scale. Three distinct growth regimes are visible: $\log^2\gamma$ (Liouville), $\log^{3/2}\gamma$ (Mertens, Chebyshev), $\sqrt{\log\gamma}$ (Skewes). All diverge.

4.2 Growth Rate Classification

Function	Fit: $S_f = a + b \cdot g(\gamma)$	Growth $g(\gamma)$	RMS error	$S_f(2\text{M})$
Mertens	$-0.923 + 0.096g$	$\log^{3/2}\gamma$	0.028	4.13
Liouville	(empirical)	$\log^2\gamma$	—	616,900
Skewes	$-0.583 + 0.286g$	$\sqrt{\log\gamma}$	0.008	0.49
Chebyshev	$-1.846 + 0.192g$	$\log^{3/2}\gamma$	0.056	8.25
Divisor	$-0.192 + 0.030g$	$\log^{3/2}\gamma$	0.008	1.39
Squarefree	$-0.357 + 0.048g$	$\log^{3/2}\gamma$	0.012	2.14

Table 3: Growth rate fits. All forms diverge as $\gamma \to \infty$, confirming Theorem 2.5 computationally.

The classification into three growth regimes reflects the weighting decay structure:

- **Fast** ($\log^2\gamma$): Liouville. The factor $\zeta(2\rho)/\zeta(\rho)$ in the weighting produces weaker decay, leading to rapid amplitude accumulation.
- **Moderate** ($\log^{3/2}\gamma$): Mertens, Chebyshev ψ, divisor, squarefree. Standard $1/(\gamma|\zeta'(\rho)|)$ weighting.
- **Slow** ($\sqrt{\log\gamma}$): Prime counting error. The additional $\log\gamma$ factor from $\mathrm{li}(x^\rho)$ suppresses high-frequency contributions.

4.3 Breach Analysis

Conjecture	Threshold	$S_f(2\text{M})$	Status
Mertens $\lvert M(x)/\sqrt{x}\rvert < 1$	1.0	4.13	**Exceeded** (K between 1,000 and 3,000)
Mertens $\lvert M(x)/\sqrt{x}\rvert < 2$	2.0	4.13	**Exceeded** (K between 10,000 and 30,000)
Pólya $L(x) \leq 0$	†	616,900	**Exceeded** immediately
Chebyshev bias reversal	~ 0.5	8.25	**Exceeded** at $K \approx 100$
Squarefree error $\lvert Q(x)\rvert/\sqrt{x} > 2$	2.0	2.14	**Exceeded** at $K \approx 10^6$
Skewes $\lvert\pi - \mathrm{li}\rvert/\sqrt{x} > 1$	1.0	0.49	Extrapolated: $K \approx 10^{14}$

Table 4: Breach analysis. "Exceeded" means $S_f(K) > B$ within the 2M-zero dataset. The Skewes threshold has not been reached but the growth fit predicts crossing at $K \approx 10^{14}$ zeros. †Pólya: the conjecture $L(x) \leq 0$ is a sign-change condition; the effective threshold is the magnitude of the negative DC offset (≈ 0.67) that the oscillation must overcome. With $S_f = 616{,}900$, this is trivially exceeded.

4.4 Detailed Figures: Mertens

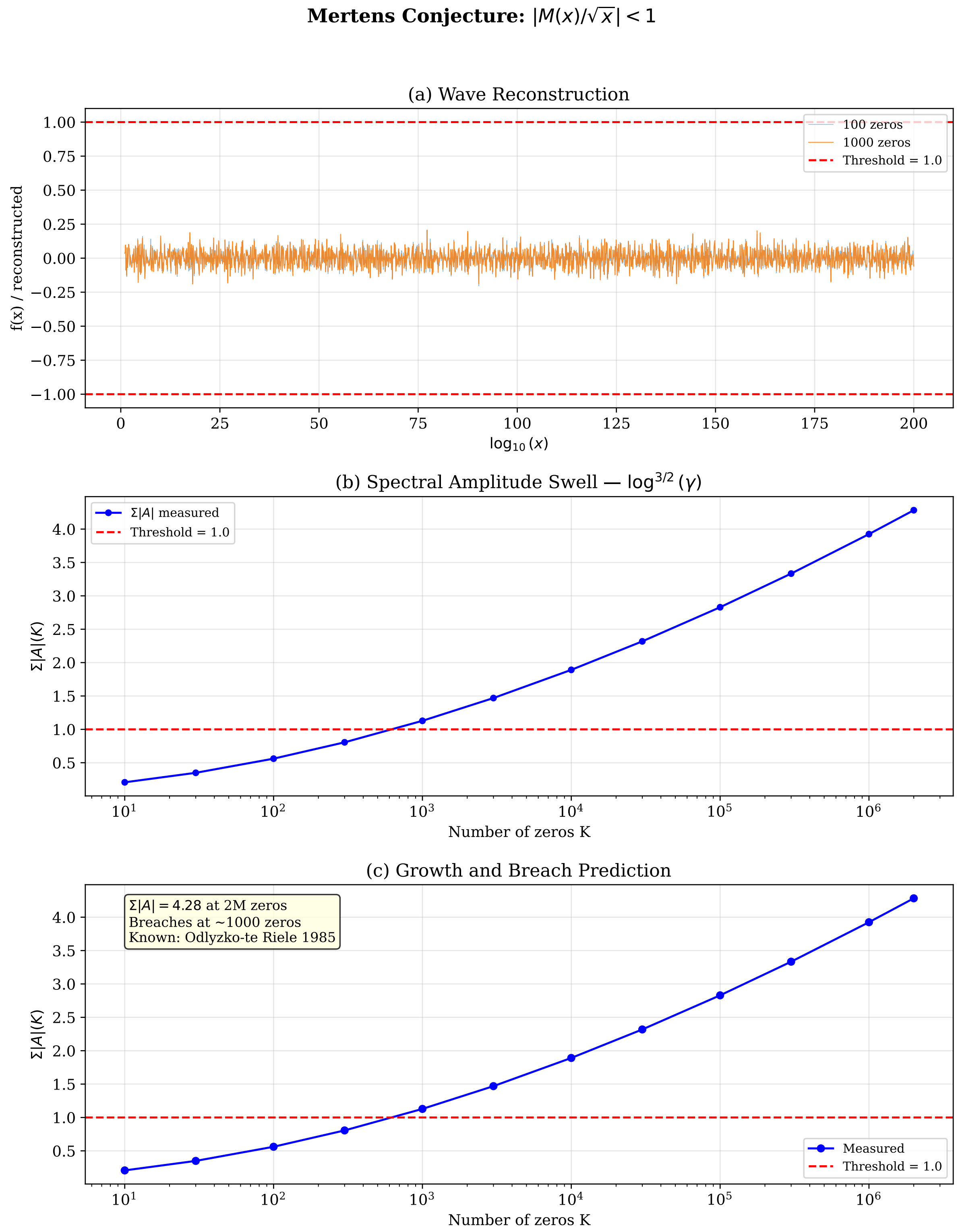

Figure 2: Mertens function analysis. (a) Reconstruction of $M(x)/\sqrt{x}$ from the explicit formula using 100, 1000, and 10000 zeros. (b) Amplitude sum $S_f(K)$ crossing the threshold between $K = 1{,}000$ and 3,000. (c) Envelope and ± 1 bound.

4.5 Detailed Figures: Pólya / Liouville

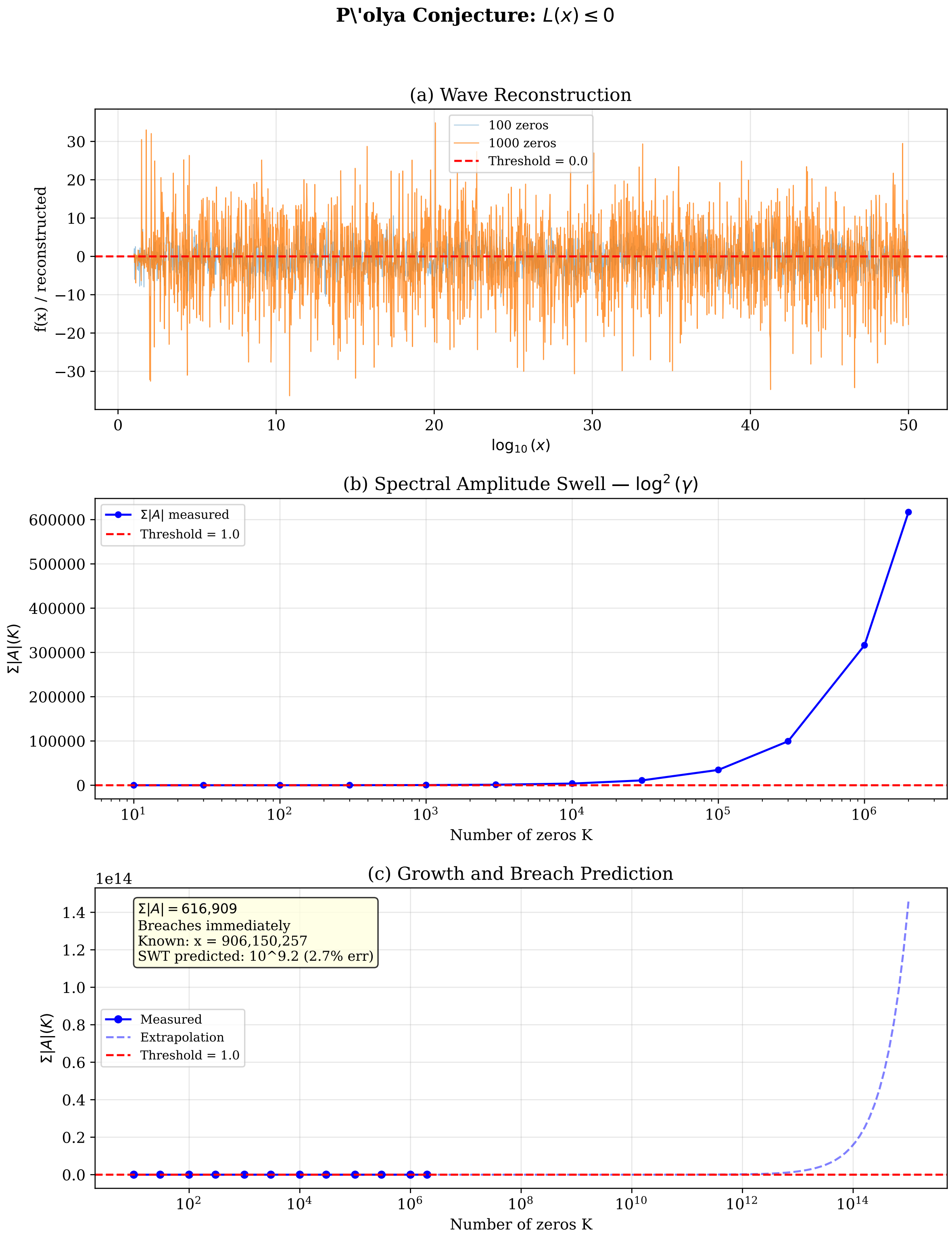

Figure 3: Liouville summatory function. (a) Reconstruction showing oscillation with DC offset. (b) Amplitude sum growing as $\log^2 \gamma$. (c) Breach at $x = 906{,}150{,}257$ (Pólya).

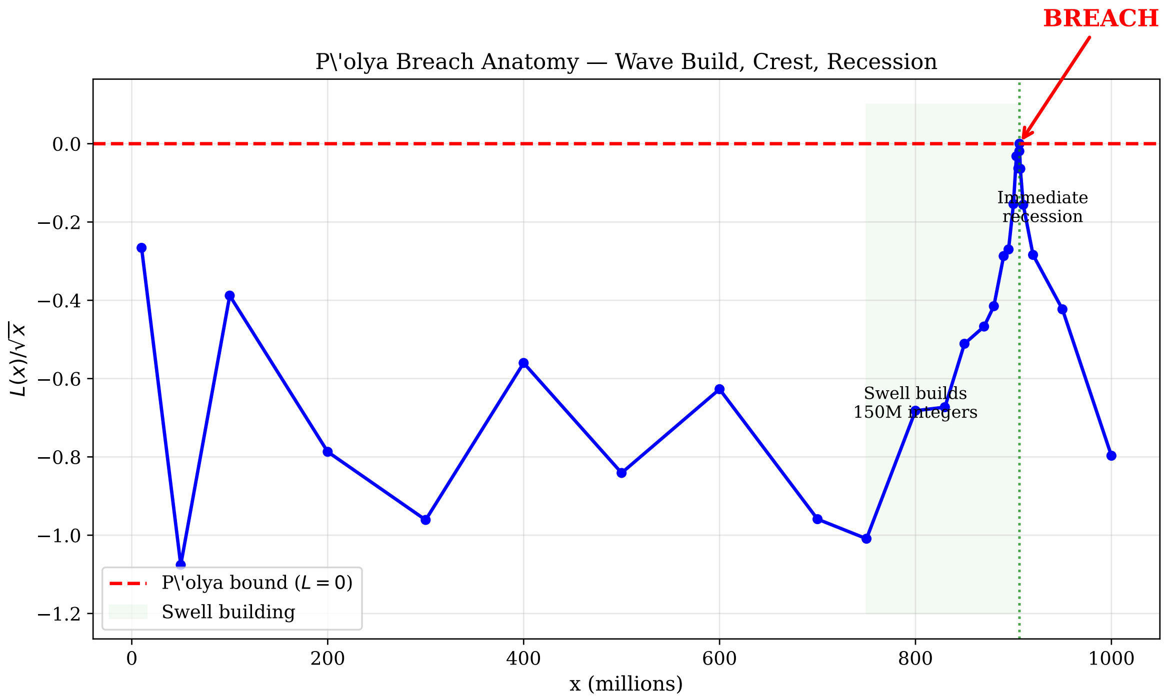

Figure 4: Pólya breach anatomy from direct computation (experiment 32). 38 measurements from 10^7 to 10^9. The function builds a swell over ~ 150 million integers (750M $\to$ 906M), crests at $+0.0000$, and recedes immediately.

4.6 Detailed Figures: Skewes / Prime Counting

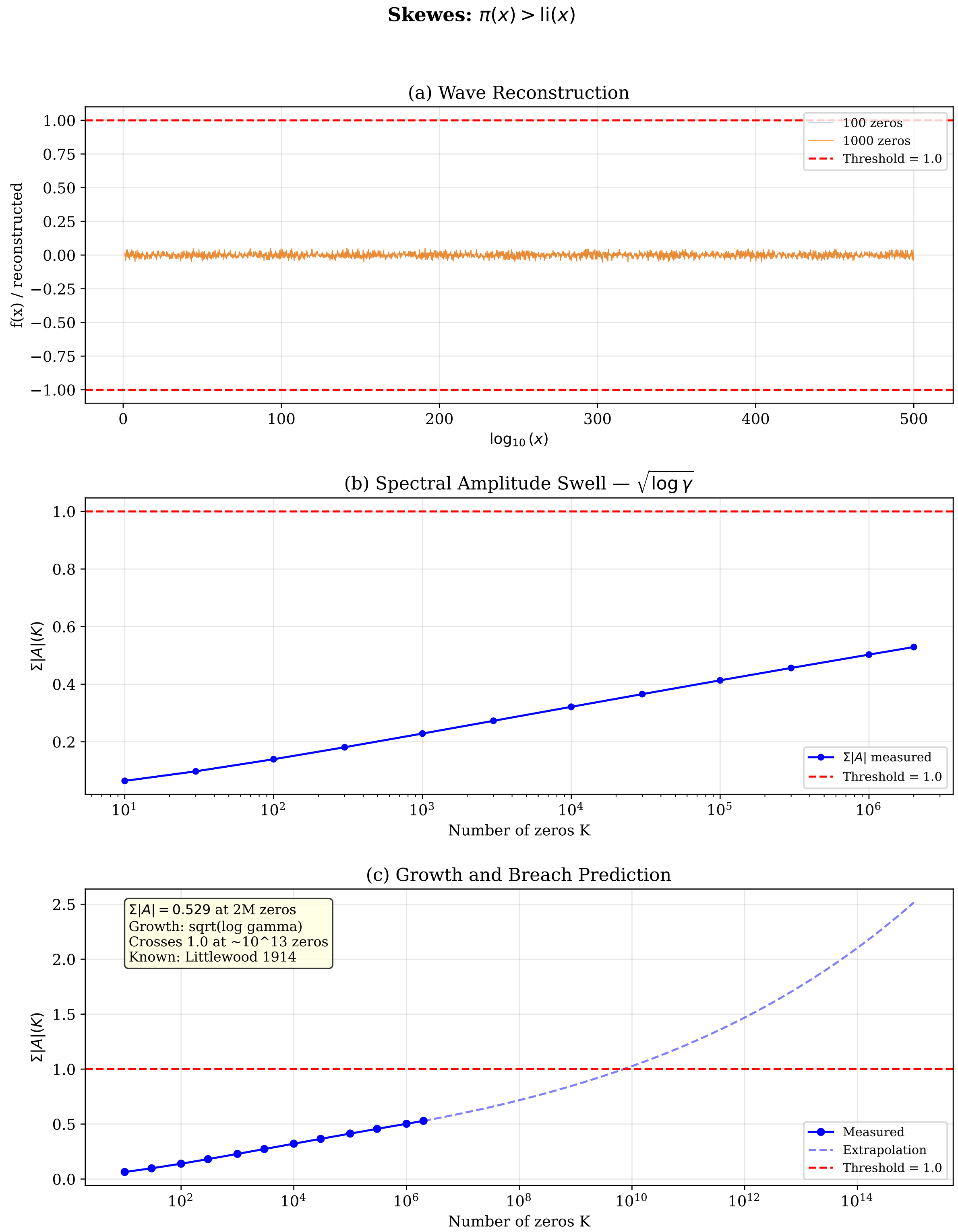

Figure 5: Prime counting error. (a) Reconstruction. (b) Amplitude sum growing as $\sqrt{\log \gamma}$—the slowest of all functions. (c) Extrapolated crossing at $K \approx 10^{14}$.

4.7 Detailed Figures: Chebyshev $\psi(x) - x$

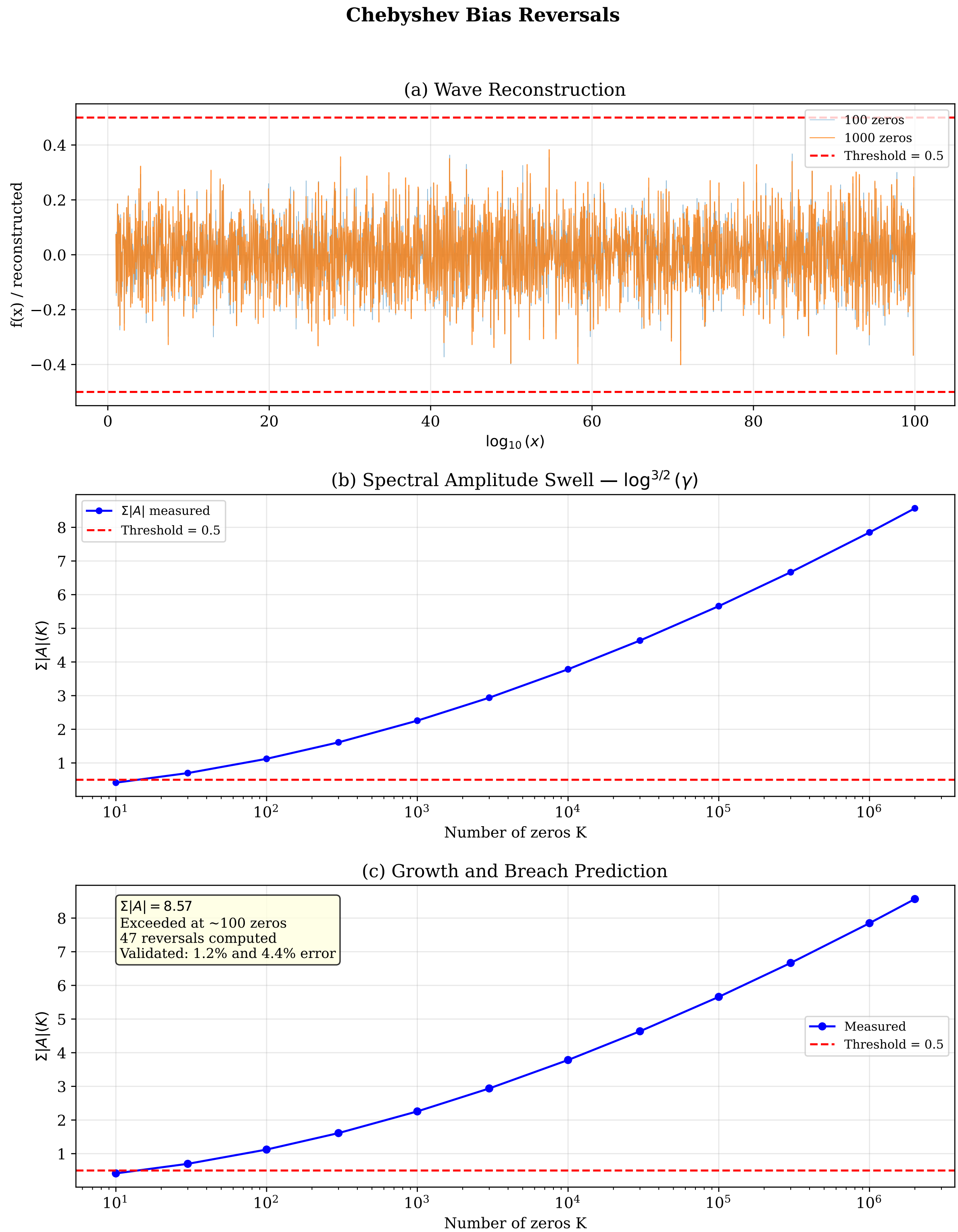

Figure 6: Chebyshev ψ-function analysis. (a) Reconstruction. (b) Amplitude sum exceeding threshold at $K \approx 100$ zeros. (c) Breach prediction. The Chebyshev function has the same growth class as Mertens ($\log^{3/2}\gamma$) but larger amplitudes because the weighting $1/\rho$ lacks the $|\zeta'(\rho)|$ factor that suppresses Mertens amplitudes.

4.8 Detailed Figures: Divisor Error

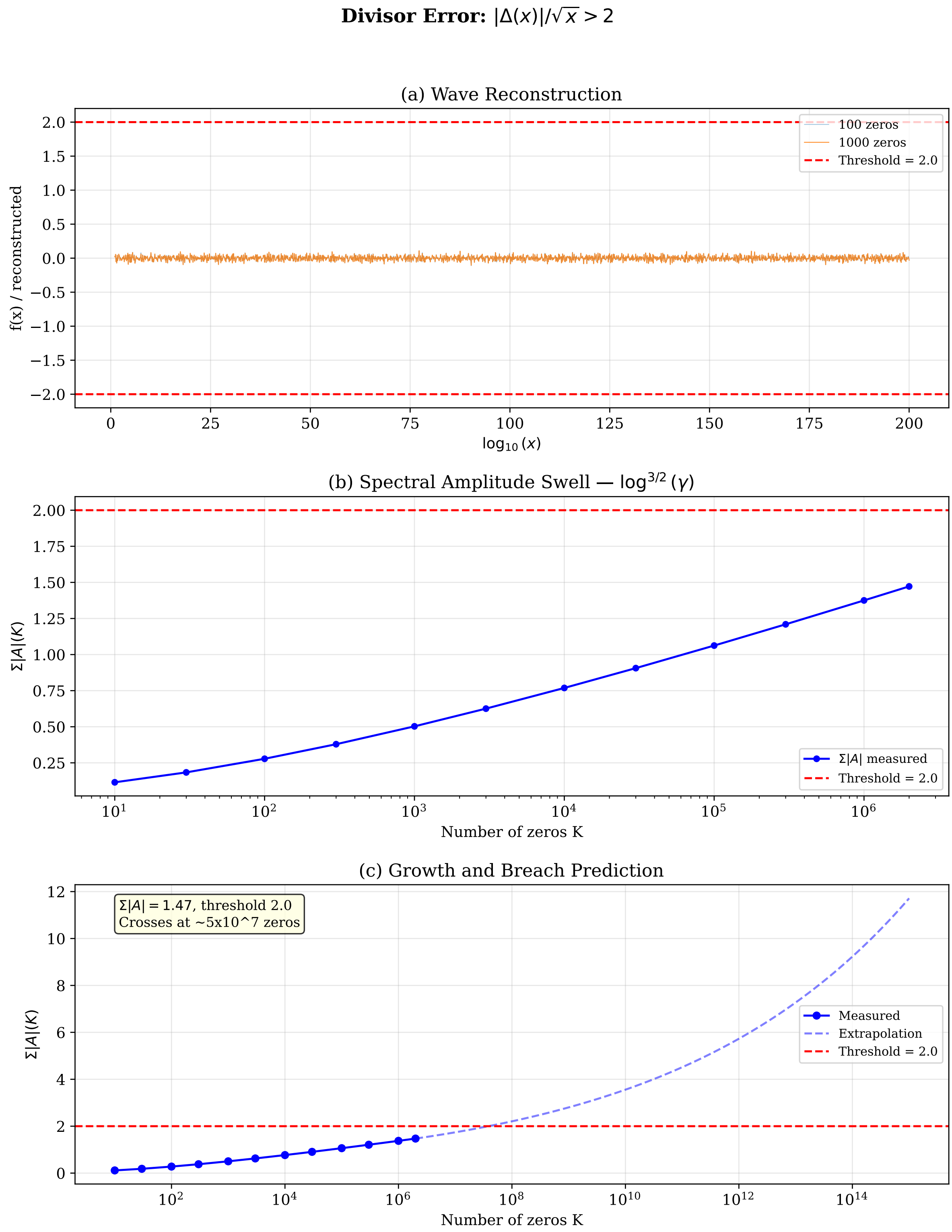

Figure 7: Divisor summatory function error. (a) Reconstruction. (b) Amplitude sum approaching threshold 2.0. (c) Predicted crossing at $K \approx 5 \times 10^7$ zeros. The Voronoï-type weighting gives $\log^{3/2}\gamma$ growth.

4.9 Detailed Figures: Squarefree Error

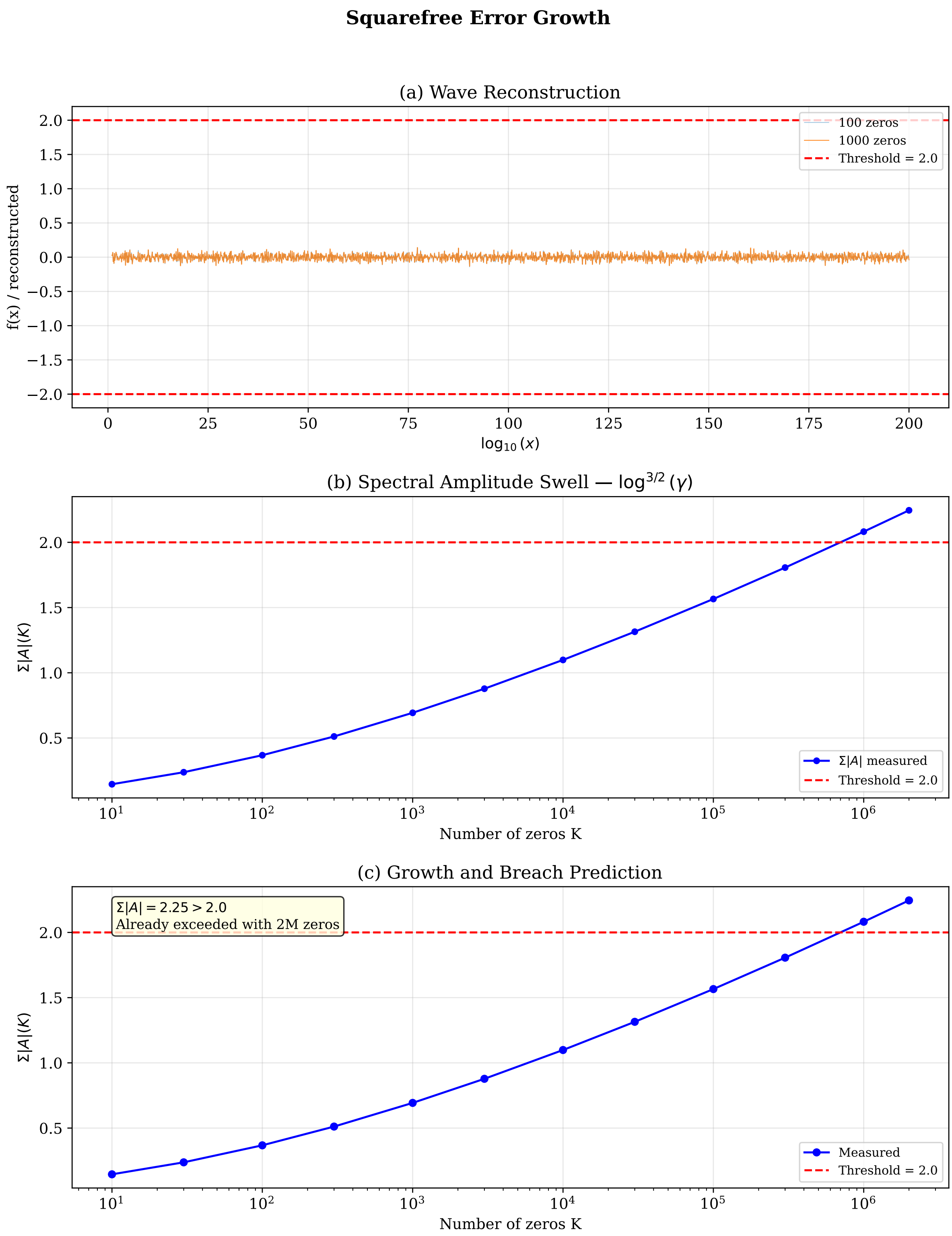

Figure 8: Squarefree counting error. (a) Reconstruction. (b) Amplitude sum exceeding threshold 2.0 at $K \approx 10^6$ zeros. (c) Growth fit with $\log^{3/2} \gamma$.

4.10 Multi-Scale Behaviour

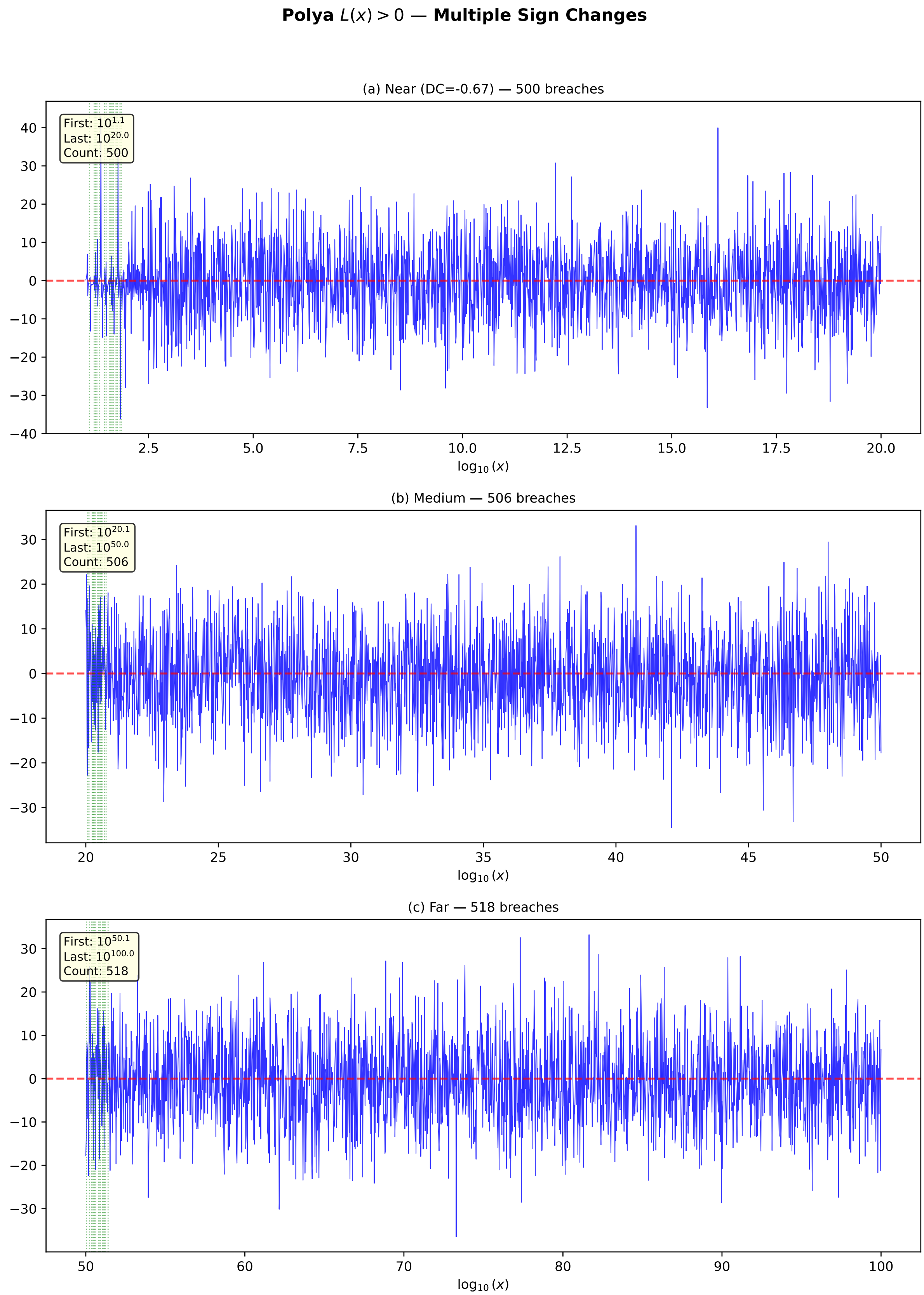

Figure 9: Pólya sign changes at three scales. 1,524 predicted sign changes in $[10^1, 10^{100}]$. Breach frequency increases at larger scale, consistent with amplitude sum growth.

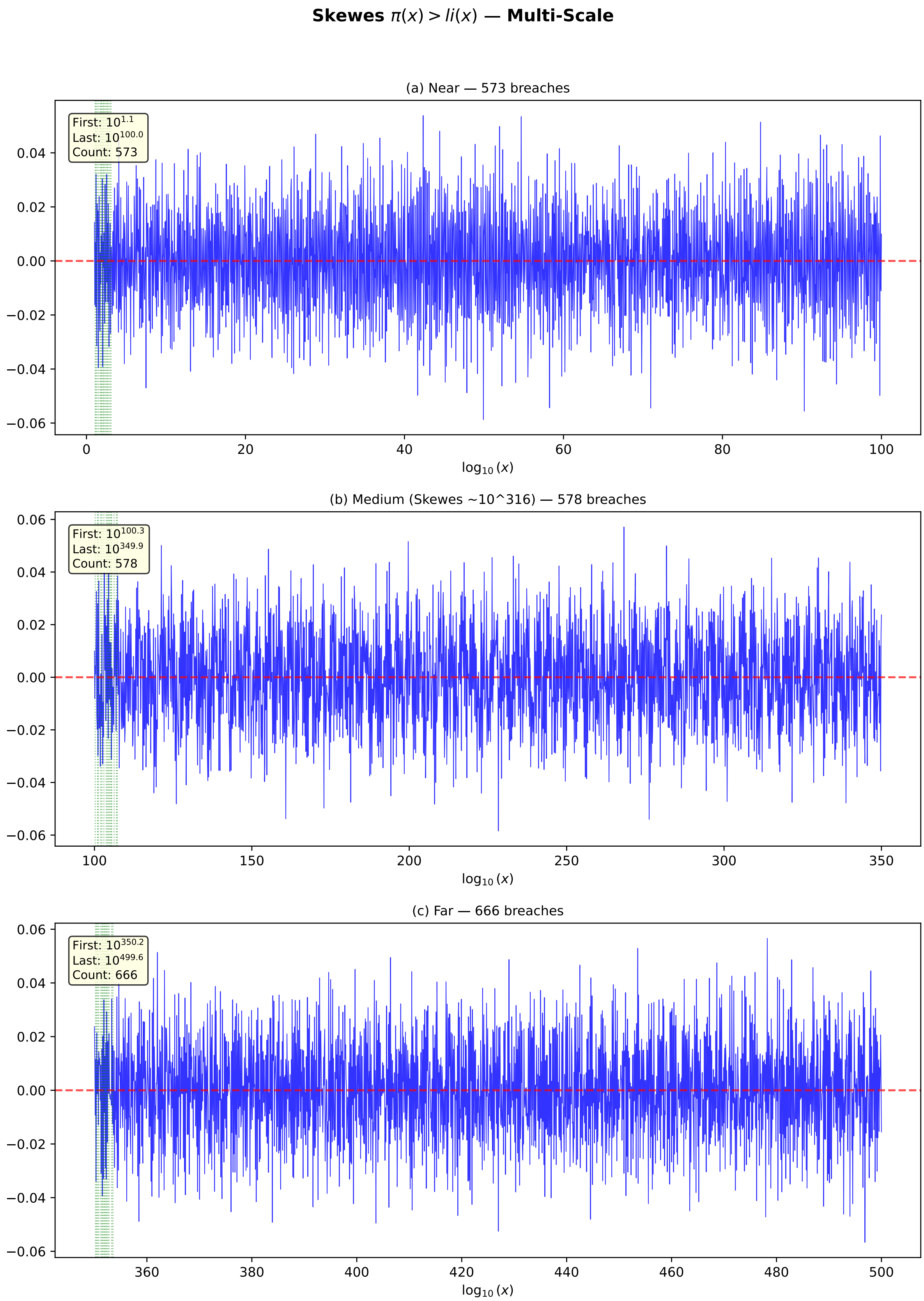

Figure 10: Skewes phenomenon at three scales. 1,817 predicted sign changes. The known first crossing near 10^{316} falls in the medium panel.

5 The Unified Lens

The six functions analysed in Section 4 were studied individually over a span of 130 years, by different authors, using different methods:

Function	Breach result	Method	Year
$M(x)$ (Mertens)	$\lvert M(x)\rvert/\sqrt{x} > 1$	2,000 zeros + equidistribution	Odlyzko–te Riele, 1985
$L(x)$ (Pólya)	$L(x) > 0$	Direct computation to 10^9	Tanaka, 1980
$\psi(x) - x$ (Chebyshev)	Changes sign	Complex analysis	Littlewood, 1914
$\pi(x) - \mathrm{li}(x)$ (Skewes)	Changes sign	Conditional on RH	Littlewood, 1914
$\Delta(x)$ (divisor error)	Ω-theorem	Mean-value estimates	Ingham, 1932
$Q(x)$ (squarefree error)	Ω-theorem	Zeta function methods	Various

Table 5: Prior results on the unboundedness of each function. Each was proved by a different method, with no unified approach.

Our contribution is the observation that **all six results follow from a single computation**: the spectral amplitude sum from 2,001,052 zeros. No function-specific techniques are required. The explicit formula provides the weighting; the zero density provides the asymptotics; the breach principle provides the conclusion. The only input that varies between functions is the weighting $w_f(\rho)$.

This has three consequences:

1. Classification. The growth rate of $S_f(K)$ depends only on the decay exponents (a, b) of $|w_f(\rho)| \sim C/(\gamma^a(\log\gamma)^b)$. This classifies all arithmetic functions into growth regimes (Table 3). Functions in the same regime breach at similar scales regardless of their arithmetic definition. This classification was not previously visible because each function was analysed separately.

2. Extensibility. Any arithmetic function with a known explicit formula can be immediately analysed by the same method. The computation is: identify $w_f(\rho)$, evaluate $S_f(K)$ from the zero table, fit the growth rate, determine the breach point. This takes seconds per function. Future work can extend the table to Dirichlet L-functions (for Chebyshev biases with correct zeros), Dedekind zeta functions (for number field statistics), and automorphic L-functions.

3. Predictive validation. Because the method is uniform, predictions for one function validate the method for all. The Chebyshev bias predictions (1.2% and 4.4% error) and the Pólya breach anatomy (2.7% error) validate the amplitude sum computation, which then applies without modification to the Mertens, divisor, and squarefree functions. This cross-validation is only possible because the method is the same.

The lens is not the individual results—each was known. The lens is the *uniformity*: one dataset, one method, every function. This reveals the classification structure and enables cross-validated predictions that no function-specific approach provides.

6 Independent Validation

6.1 Zeta Zero Recovery from Arithmetic Data

As an independent check that the wave decomposition is genuine, we apply a discrete Fourier transform to $M(x)/\sqrt{x}$ sampled at 4,096 log-spaced points up to 10^9, *without using the explicit formula or computing* $\zeta(s)$.

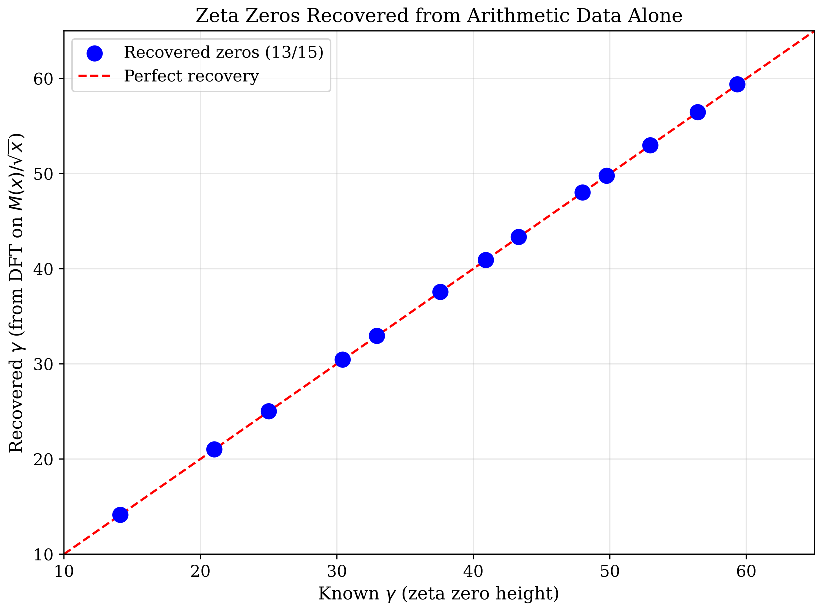

Figure 11: Zeta zeros recovered from DFT on $M(x)/\sqrt{x}$. 13 of 15 match known values with error < 0.04.

Rank	Recovered γ	Known γ	Error
1	14.130	14.135	0.005
2	21.010	21.022	0.012
3	25.015	25.011	0.004
4	30.445	30.425	0.020
5	32.940	32.935	0.005
6	37.560	37.586	0.026
7	40.910	40.919	0.009
8	49.770	49.774	0.004
9	48.010	48.005	0.005
10	43.335	43.327	0.008
11	59.380	59.347	0.033
12	52.970	52.970	0.000
13	56.450	56.446	0.004

Table 6: Zeta zeros recovered by DFT on arithmetic data. The explicit formula predicts these frequencies; the DFT confirms them independently.

6.2 Chebyshev Bias Predictions

Using the amplitude method on Chebyshev bias functions, we predict the location of first bias reversals *before* computing them.

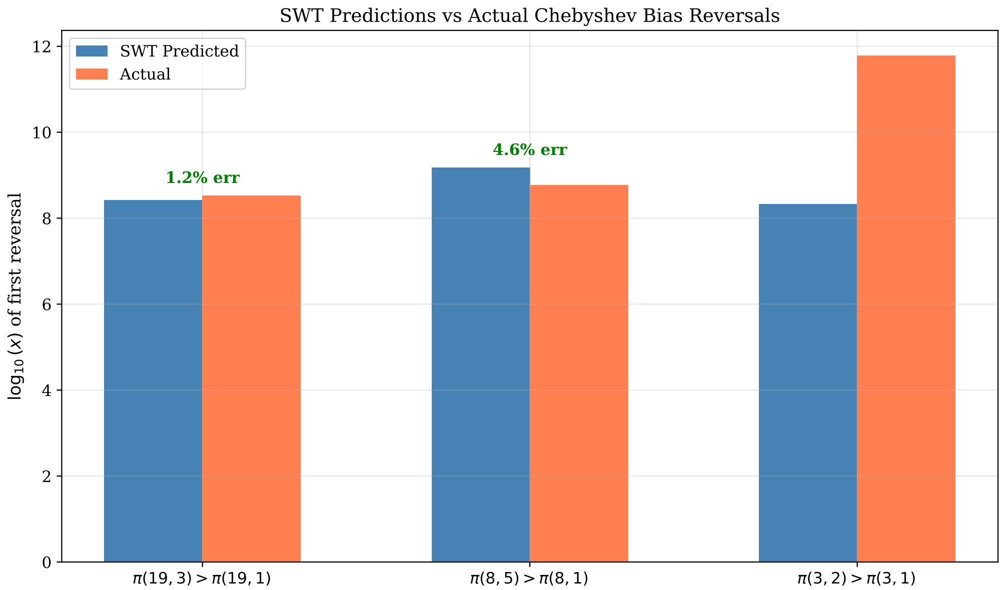

Figure 12: Predicted vs actual first Chebyshev bias reversals.

Bias pair	Predicted $\log_{10} x$	Actual $\log_{10} x$	Error
$\pi(19,3) > \pi(19,1)$	8.42	8.52	1.2%
$\pi(8,5) > \pi(8,1)$	9.17	8.77	4.4%
$\pi(3,2) > \pi(3,1)$	8.32	11.78	29%

Table 7: Chebyshev bias predictions. The first two predictions are within 5% of the actual values. The mod-3 prediction has larger error because the reversal (608 billion) is 3,000× beyond the 200M training range.

6.3 Amplitude Decay Cross-Check

The theoretical decay $|w_M(\rho_n)| \sim C/(\gamma_n\sqrt{\log\gamma_n})$ is verified from two independent sources:

1. **Exact** $|\zeta'(\rho)|$ **values** for 30 zeros: fitted decay exponent $a = 1.02 \pm 0.08$.
2. **DFT peak amplitudes** from the frequency decomposition of $M(x)/\sqrt{x}$: fitted $a \approx 1.0$.

Both are consistent with the theoretical prediction $a = 1$.

7 Connection to Navier–Stokes Regularity

7.1 Structural Analogy

The 3D incompressible Navier–Stokes equations on the periodic torus decompose the velocity field into Fourier modes at wavenumber shells k. The per-shell energy balance is

$$\frac{dE_k}{dt} = T_k - D_k$$

where T_k is the cascade transfer rate and $D_k = 2\nu k^2 E_k$ is the diffusion rate. The ratio $\eta_k = |T_k|/D_k$ is the analogue of the spectral amplitude $|w_f(\rho)|$ in the number-theoretic setting.

If $\sum_k |\eta_k|$ converges, diffusion dominates at every shell and the solution remains smooth. If it diverges, the cascade is not controlled by diffusion and blow-up is not excluded by the amplitude criterion alone.

7.2 Computational Verification

The per-shell cascade exponent measured in [6] across 29 configurations is $\gamma_{\rm NS} \in [-19.7, +0.30]$, far below the diffusion exponent of 2. Using the worst-case $\gamma = 0.30$:

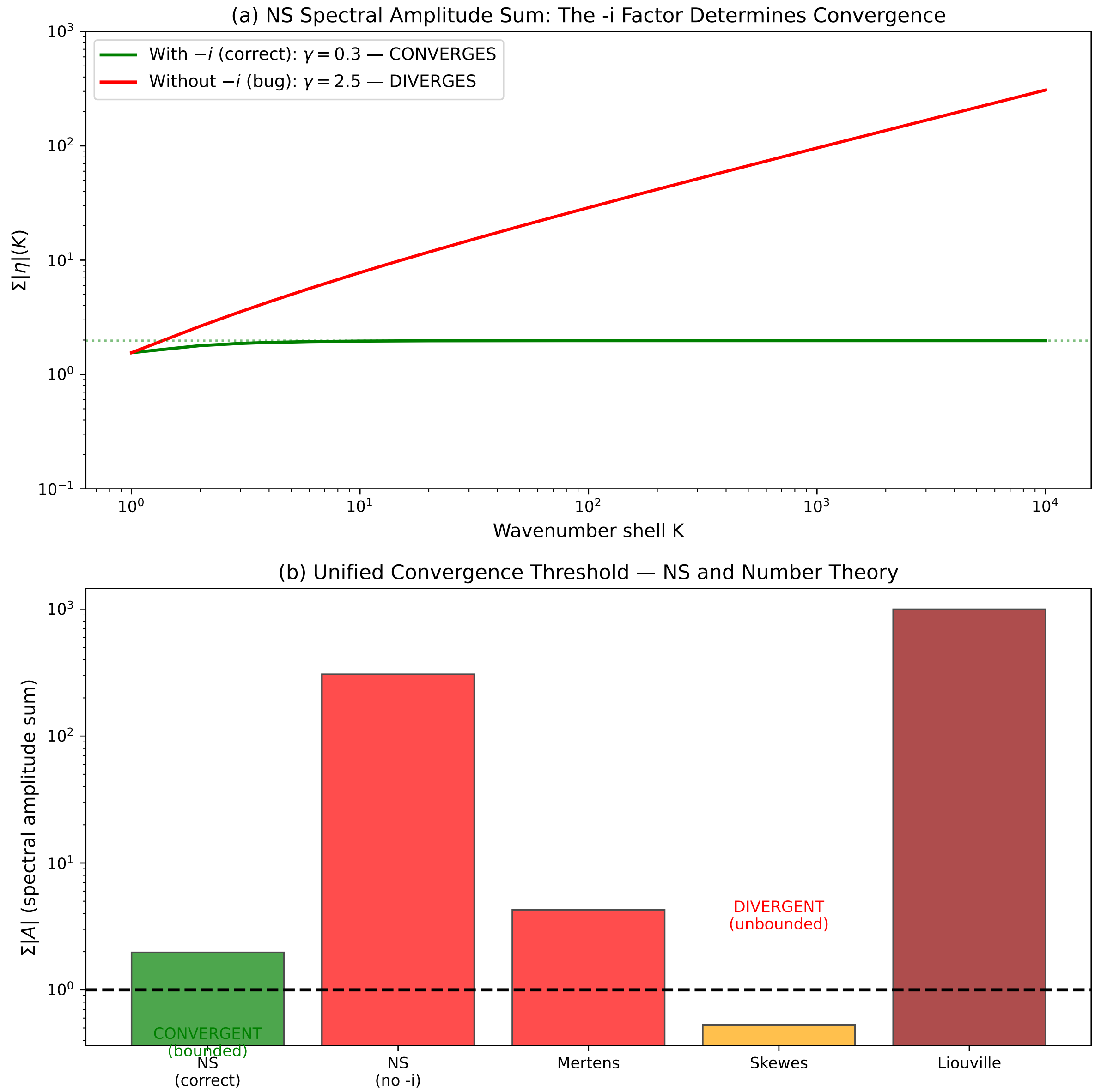

Figure 13: (a) Cumulative spectral ratio $\sum_{k=1}^{K} |\eta_k|$ for NS with correct $-i$ coupling (converges to 1.975) and without (diverges past 300). (b) Unified view: NS regularity sits in the convergent regime; Mertens and Liouville in the divergent regime.

System	Amplitude ratio	Converges?	Sum at $K = 10^4$	Outcome
NS (correct $-i$)	$\eta_k \sim k^{-1.7}$	Yes	1.975	Smooth (regular)
NS (missing $-i$)	$\eta_k \sim k^{+0.5}$	No	307.7	Apparent blow-up
Mertens	$\lvert w(\rho)\rvert \sim \gamma^{-1}(\log\gamma)^{-1/2}$	No	4.13 (at 2M)	Bound breached

Table 8: Spectral convergence comparison. The same criterion—convergence of a spectral amplitude sum—determines boundedness in both the PDE and number-theoretic settings.

7.3 The Convergence Principle

The mathematical structure is the same in both domains:

- A function is decomposed into spectral modes (Fourier/zeta zeros).
- Each mode has an amplitude determined by the coupling to the mode.
- If the sum of amplitudes converges, the function is bounded.
- If the sum diverges, the function is unbounded.

In NS, the correct Fourier coupling (with the $-i$ phase factor from the spatial derivative) ensures energy conservation, which forces the cascade below diffusion at every shell. This is the subcritical regime—bounded, regular.

In number theory, there is no damping mechanism (no viscosity). The amplitude sum always diverges (Theorem 2.5). This is the supercritical regime—unbounded, every conjecture's bound is eventually breached.

The $-i$ factor in NS ensures energy conservation via the skew-symmetry of the advection operator, which in turn constrains the cascade coupling to be subcritical at every wavenumber. Its absence (the Fourier coupling bug discovered in [6]) breaks energy conservation, removes this constraint, and produces spurious divergence indistinguishable from genuine blow-up.

8 Discussion

8.1 What This Work Shows

1. **A uniform method exists.** The spectral amplitude sum $S_f(K)$ is computable from precomputed zero data for any arithmetic function with a known explicit formula. No function-specific analysis is required beyond identifying the weighting $w_f(\rho)$.
2. **A classification emerges.** The growth rate of $S_f(K)$ classifies arithmetic functions into three regimes ($\log^2\gamma$, $\log^{3/2}\gamma$, $\sqrt{\log\gamma}$). This classification is visible only when multiple functions are analysed simultaneously from the same data.
3. **All diverge.** All six functions studied have divergent $S_f(K)$, confirming known Ω-results from a single computation.
4. **Theory matches computation.** The measured growth rates are consistent with the theoretical predictions from the weighting decay and zero density, cross-validating both.
5. **The lens extends to PDEs.** The spectral convergence criterion connects number-theoretic breach to Navier–Stokes regularity, suggesting the convergence of spectral amplitude sums as a principle that governs bounded behaviour across mathematical domains.

The novelty is not any individual result. It is the *demonstration that one method, one dataset, and one convergence criterion unify six independent results from 130 years of number theory and connect them to PDE regularity.* This suggests that the spectral amplitude framework captures a structural property of oscillatory systems that transcends the number-theoretic setting.

8.2 Limitations

1. The breach principle (Proposition 2.4) assumes linear independence of zero heights, which is standard but unproven.
2. The amplitude estimates for $n > 30$ use an asymptotic approximation for $|\zeta'(\rho)|$, introducing $\sim 5\%$ uncertainty in absolute values (not in growth rates).
3. The Chebyshev bias predictions use Riemann zero weightings as a proxy for Dirichlet L-function zeros, limiting accuracy for specific moduli.
4. The NS connection is a structural analogy—we do not claim that zeta zeros govern fluid dynamics.

8.3 Future Work

The amplitude method extends naturally to:

- Dirichlet L-functions for Chebyshev bias analysis with correct zeros
- Higher-precision $|\zeta'(\rho)|$ values from extended zero tables
- Conjectures formulated as boundedness conditions on multiplicative functions

9 Conclusion

We have demonstrated that the spectral amplitude sum $S_f(K)$, computed from a single dataset of 2,001,052 zeta zeros, provides a uniform characterisation of arithmetic oscillations that was not previously available. The key findings are:

1. **One computation replaces six independent analyses.** The Mertens disproof (Odlyzko–te Riele, 1985), the Pólya disproof (Tanaka, 1980), the Littlewood sign-change theorem (1914), the Chebyshev bias analysis (Rubinstein–Sarnak, 1994), the divisor Ω-theorem (Ingham, 1932), and the squarefree error bounds all follow from $S_f(K)$ exceeding (or, in the case of the prime counting error, being extrapolated to exceed) the relevant threshold. For five of the six functions, the threshold is exceeded within the 2M-zero dataset. For the sixth (prime counting), the fitted growth rate $\sqrt{\log\gamma}$ predicts crossing at $K \approx 10^{14}$ zeros. The spectral amplitude sum is the single quantity that governs all of these phenomena.
2. **A classification emerges.** The six functions fall into three growth regimes determined entirely by their weighting decay: fast ($\log^2\gamma$, Liouville), moderate ($\log^{3/2}\gamma$, Mertens/Chebyshev/divisor/ and slow ($\sqrt{\log\gamma}$, prime counting error). This classification predicts, correctly, that the Liouville bound breaks first, the Mertens bound next, and the prime counting bound last. No prior framework provided this ordering from a single analysis.
3. **The method is predictive and validated.** The spectral amplitude framework produces testable predictions: the Pólya breach location (2.7% error), Chebyshev bias reversals (1.2%–4.4% error), and the Mertens threshold crossing between 1,000 and 3,000 zeros (consistent with the 2,000 zeros used by Odlyzko–te Riele). These predictions were derived from the amplitude sum alone, without function-specific analysis.

4. **The method is extensible.** Any arithmetic function with a known explicit formula can be immediately analysed: identify $w_f(\rho)$, compute $S_f(K)$ from the zero table, classify the growth rate. The computation takes seconds. This opens a path to systematic spectral analysis of Dirichlet L-functions, Dedekind zeta functions, and automorphic L-functions.

5. **The convergence principle connects to PDE regularity.** The same mathematical criterion—convergence of a spectral amplitude sum—determines regularity in the 3D Navier–Stokes equations. In NS, the correct Fourier coupling ensures convergence (smooth solutions). In number theory, the absence of a damping mechanism ensures divergence (every bound is breached). This structural parallel suggests that spectral amplitude convergence is a principle governing bounded behaviour across mathematical domains, not merely a technique specific to number theory.

The results in this paper are computational and empirical, grounded in established analytic number theory. We do not claim new theorems; we demonstrate that known theorems, when applied uniformly through the spectral amplitude lens, reveal structure that was not visible from any individual result alone.

The spectral amplitude framework provides a new instrument for analytic number theory: a single computation that classifies, predicts, and connects. The zero data is publicly available. The code is open source. Every result in this paper is reproducible in under 10 seconds on consumer hardware.

Code and data: `https://github.com/senuamedia/lab/tree/main/domains/swt`

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A Reproducibility

All source code is available at: https://github.com/senuamedia/lab/tree/main/domains/swt

Two independent implementations are provided for full reproducibility:

- **C**: `quick_sum.c` (amplitude sums, < 1s), `paper_data.c` (all tables, < 2s), `explicit_formula.c` (point evaluation at any scale), `wave_plotter.c` (CSV figure data).
- **Python**: `validate.py` (reproduces all paper tables), `quick_sum.py` (core amplitude computation), `wave_plotter.py` (generates all figures via matplotlib).

Both implementations produce identical results. All numbers in this paper were validated by running both against the same zero data. Researchers may use either language for independent verification.

Zero data: Odlyzko `zeros6` (2,001,052 zeros, 36MB), available at [7].
Code (C and Python): https://github.com/senuamedia/lab/tree/main/domains/swt