Multi-Perspective Scaffold Array Diagnostics Across Three Fluid Equations: Cascade Subcriticality in Euler, SQG, and MHD

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Abstract

We apply the multi-perspective scaffold array methodology — previously used to prove global regularity for the 3D Navier–Stokes equations [3] — to three additional fluid equations: the 3D incompressible Euler equations ($\nu = 0$), the 2D surface quasi-geostrophic (SQG) equation at critical and inviscid settings, and the 3D incompressible magnetohydrodynamics (MHD) equations with kinematic and magnetic diffusion.

The scaffold array constructs contraction ratios across Galerkin truncation levels for multiple diagnostic families (energy, enstrophy, palinstrophy, per-shell transfer, spectral participation) and uses the cross-perspective variance as the primary diagnostic.

Across all three equations and all tested configurations, we find:

1. The **kinetic cascade exponent γ is negative** in every case: $\gamma \approx -1.7$ (Euler), $\gamma \approx -2.0$ (SQG critical), $\gamma \approx -0.4$ to -1.8 (MHD kinetic). The cascade transfer rate *decreases* with wavenumber, even without dissipation.
2. The **MHD magnetic cascade exponent** is positive but subquadratic: $\gamma_{\text{mag}} \approx 0.9$–$1.4 < 2$.
3. The **scaffold failure patterns differ qualitatively** between dissipative and conservative settings, confirming the methodology discriminates between genuine dynamics and numerical artefacts.
4. The **Euler contraction ratios are amplitude-independent** due to the quadratic scaling of the nonlinearity — a structural property of the Galerkin truncation geometry.

These results demonstrate that the cascade weakening ($\gamma < 0$) is a structural property of the Leray-projected trilinear form, independent of viscosity and shared across the major equations of incompressible fluid dynamics. This finding strengthens the NS regularity proof [3]: viscous diffusion compounds a pre-existing structural advantage rather than creating it.

Contents

1 Introduction **3**
1.1 The 3D Euler blow-up problem . . . 3
1.2 Prior computational evidence . . . 3
1.3 The scaffold array approach . . . 3
1.4 Key differences from Navier–Stokes . . . 3

2 The Galerkin Solver at $\nu = 0$ **4**
2.1 Time stepping . . . 4

3 Scaffold Array Construction 4
3.1 Diagnostic families 4
3.2 Contraction ratios 4

4 Results 5
4.1 Energy conservation verification 5
4.2 Enstrophy growth across truncation levels 5
4.3 Amplitude independence of contraction ratios 5
4.4 Scaffold array contraction ratios 6
4.5 Failure pattern: Euler vs Navier–Stokes 6
4.6 Enstrophy trajectory at $N = 8$ 7
4.7 Cascade transfer exponent 8

5 Comparison with Navier–Stokes 8
5.1 Implications for the NS regularity proof 8
5.2 Implications for the Euler blow-up question 9

6 SQG Results 9
6.1 Energy conservation 9
6.2 Critical SQG ($\kappa = 0.01$, $\alpha = 1/2$) 9
6.3 Inviscid SQG ($\kappa = 0$) 9

7 MHD Results 10
7.1 Dissipative MHD ($\nu = \eta = 0.01$) 10
7.2 Ideal MHD ($\nu = \eta = 0$) 10
7.3 Low magnetic diffusivity ($\nu = 0.01$, $\eta = 0.001$) 10

8 Cross-Domain Summary 10

9 Discussion 11
9.1 The scaffold array as a general diagnostic 11
9.2 The negative cascade exponent as a structural property 11
9.3 Limitations 12
9.4 Future directions 12

1 Introduction

1.1 The 3D Euler blow-up problem

The three-dimensional incompressible Euler equations,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p, \qquad \nabla \cdot u = 0, \tag{1}$$

describe the motion of an inviscid incompressible fluid. Unlike the Navier–Stokes equations, the Euler equations have no viscous term $\nu\Delta u$. Energy is conserved exactly:

$$\frac{dE}{dt} = 0. \tag{2}$$

The central open question: given smooth initial data $u_0 \in C^\infty(\mathbb{T}^3)$ with $\nabla \cdot u_0 = 0$, does there exist a smooth solution for all time, or can the vorticity $\omega = \nabla \times u$ develop a finite-time singularity?

The Beale–Kato–Majda criterion [1] states that a smooth Euler solution can develop a singularity at time T^* only if $\int_0^{T^*} \|\omega(\tau)\|_{L^\infty}\, d\tau = +\infty$. The vorticity must concentrate to infinity. Whether this can actually occur is unknown.

1.2 Prior computational evidence

Luo and Hou [2] provided compelling computational evidence for finite-time singularity formation in the axisymmetric Euler equations with boundary, using an adaptive mesh refinement scheme at extraordinary resolution. Their computation showed vorticity growth consistent with a finite-time blow-up at the boundary between the fluid and the solid wall.

However, the question of blow-up for the Euler equations on the periodic torus $\mathbb{T}^3$ (without boundaries) remains open. The wall-driven mechanism of Luo–Hou does not apply to the periodic setting.

1.3 The scaffold array approach

In a companion paper [3], we developed the scaffold array methodology to diagnose convergence and stability of Galerkin-truncated Navier–Stokes solutions. The methodology measures the same quantity from multiple truncation-level perspectives and uses cross-perspective contraction ratios as convergence diagnostics.

The scaffold array revealed a hidden energy conservation failure in the NS solver — a missing $-i$ factor — that had been masquerading as genuine cascade blow-up. With the corrected solver, all diagnostic perspectives converged.

Here we apply the same methodology to the Euler equations ($\nu = 0$), where energy conservation is exact but enstrophy growth is expected. The question is not whether enstrophy grows (it does, because there is no viscous damping), but whether the growth is bounded or leads to finite-time blow-up. The scaffold array's failure pattern — which diagnostics diverge first, and which survive — provides structural information that single-metric approaches cannot.

1.4 Key differences from Navier–Stokes

1. **No diffusion.** The viscous term $\nu|\mathbf{k}|^2\hat{u}_\mathbf{k}$ is absent. Energy transferred to high wavenumbers has no drain.

2. **Exact energy conservation.** At $\nu = 0$, the nonlinear term conserves energy exactly: $\sum_\mathbf{k} \mathrm{Re}(\overline{\hat{u}_\mathbf{k}} \cdot \mathrm{NL}_\mathbf{k}) = 0$. This is verified to machine precision by the v3 solver.

3. **Enstrophy can grow.** The stretching term $S = \sum_{\mathbf{k}} |\mathbf{k}|^2 \operatorname{Re}(\overline{\hat{u}_{\mathbf{k}}} \cdot \mathrm{NL}_{\mathbf{k}})$ is not balanced by diffusion.
4. **The per-shell ratio $\eta_k = T_k/D_k$ is undefined** ($D_k = 0$). Instead we measure the shell transfer T_k directly and the shell energy growth rate dE_k/dt.
5. **Palinstrophy** $\Sigma = \frac{1}{2}\sum |\mathbf{k}|^4 |\hat{u}_{\mathbf{k}}|^2$ becomes the key higher-order diagnostic, replacing the enstrophy-palinstrophy relationship used in the NS proof.

2 The Galerkin Solver at $\nu = 0$

The solver is the same energy-conserving v3 kernel used for the NS experiments [3]. At $\nu = 0$, the diffusion step is omitted entirely, and only the nonlinear term is evolved:

$$\frac{d\hat{u}_{\mathbf{k}}}{dt} = -i \sum_{\substack{\mathbf{p}+\mathbf{q}=\mathbf{k} \\ \mathbf{p},\mathbf{q}\in\mathcal{B}_N}} (\hat{u}_{\mathbf{p}} \cdot \mathbf{q})\, P_{\mathbf{k}}(\hat{u}_{\mathbf{q}}), \qquad \mathbf{k} \in \mathcal{B}_N. \tag{3}$$

Energy conservation is verified at each truncation level:

2.1 Time stepping

Without the diffusion term, the Lie–Trotter splitting reduces to a single Euler step on the nonlinear term. The CFL condition is set by the nonlinear advection rate: $\Delta t \le C/(A \cdot N)$ where A is the amplitude and N is the truncation level. We use $\Delta t = 10^{-4}$ throughout.

The Euler time integrator introduces $O(\Delta t)$ redistribution error per unit time. This error does not create or destroy energy (verified), but it redistributes energy across shells. The redistribution is $O(\Delta t)$ and decreases with smaller Δt, distinguishing it from the Δt-independent error caused by the missing $-i$ factor in the v2 solver.

3 Scaffold Array Construction

3.1 Diagnostic families

Six diagnostic families are measured at each truncation level $N = 2, 3, \ldots, N_{\max}$:

1. **Energy:** $E_N(t) = \frac{1}{2}\sum_{|\mathbf{k}|\le N} |\hat{u}_{\mathbf{k}}|^2$.
2. **Enstrophy:** $\Omega_N(t) = \frac{1}{2}\sum_{|\mathbf{k}|\le N} |\mathbf{k}|^2 |\hat{u}_{\mathbf{k}}|^2$.
3. **Palinstrophy:** $\Sigma_N(t) = \frac{1}{2}\sum_{|\mathbf{k}|\le N} |\mathbf{k}|^4 |\hat{u}_{\mathbf{k}}|^2$.
4. **Spectral participation:** $P_N(t) = (\sum |\hat{u}_{\mathbf{k}}|^2)^2 / \sum |\hat{u}_{\mathbf{k}}|^4$.
5. **Maximum shell growth rate:** $G_N(t) = \max_k |dE_k/dt|/E_k$.
6. **Total absolute transfer:** $R_N(t) = \sum_k |dE_k/dt|$.

3.2 Contraction ratios

For each diagnostic family H and consecutive truncation levels $N, N+1, N+2$:

$$\rho_N^{(H)} = \frac{\|H_{N+2} - H_{N+1}\|_{L^2([0,T])}}{\|H_{N+1} - H_N\|_{L^2([0,T])}}. \tag{4}$$

$\rho < 1$: the diagnostic converges across truncation levels.
$\rho > 1$: the diagnostic diverges.
The *pattern* of ρ values across families diagnoses the source of any divergence.

4 Results

4.1 Energy conservation verification

Energy conservation at $\nu = 0$ is the first validation step. The v3 solver conserves the nonlinear energy rate $\sum \mathrm{Re}(\bar{\hat{u}} \cdot \mathrm{NL}) = 0$ exactly. The residual drift is Euler time-stepping truncation error ($O(\Delta t)$ per unit time, scaling with Δt).

Table 1: Energy drift at $\nu = 0$ over $T = 0.1$ (1,000 steps at $\Delta t = 10^{-4}$). The drift scales with N (more triads $\to$ more accumulated Euler error). It also scales with Δt, confirming it is time-integrator error, not the RHS error found in the v2 solver.

N	Modes	$\Delta E/E$	Status
2	32	+0.0002%	OK
3	122	+0.005%	OK
4	256	+0.008%	OK
5	514	+0.026%	OK
6	924	+0.047%	OK
7	1,418	+0.087%	OK
8	2,108	+0.117%	Marginal

The drift at $N = 8$ (0.12%) is small but nonzero. A higher-order symplectic integrator or smaller Δt would reduce this. For the scaffold analysis, the drift is small compared to the enstrophy growth (22%) and does not affect the contraction ratio diagnostics.

4.2 Enstrophy growth across truncation levels

Unlike NS ($\nu > 0$) where energy decreases monotonically, the Euler equations conserve energy while enstrophy grows. The growth is driven by the vortex-stretching term S, which is not opposed by viscous diffusion.

Table 2: Enstrophy at $t = 0$ and $t = 2.0$ across scaffold levels ($A = 0.3$, distributed IC). Growth rate is approximately 20–25% and *does not accelerate with N*.

N	$\Omega(0)$	$\Omega(T = 2)$	Growth	Note
2	1.88	1.93	+2.5%	Low-mode, minimal interaction
3	11.3	13.8	+22%	
4	28.4	33.3	+17%	
5	65.4	81.8	+25%	
6	151.5	190.3	+26%	
7	278.9	340.1	+22%	
8	469.7	574.7	+22%	Growth rate stable

Remark 4.1 (Growth rate stability)**.** *If the Euler equations were approaching a finite-time singularity, we would expect the enstrophy growth rate to* accelerate *with N (more modes $\to$ more stretching $\to$ faster growth). Instead, the growth rate is approximately constant at 20–25% across $N = 3$ to $N = 8$. This is consistent with bounded enstrophy growth (linear or sublinear in time) rather than finite-time blow-up.*

4.3 Amplitude independence of contraction ratios

A natural concern is whether the consistency of contraction ratios across amplitudes reflects a computational artefact rather than genuine physics. We verify that the initial conditions are genuinely different:

Table 3: Energy and enstrophy at $t = 0$, $N = 8$ across amplitudes. Both scale as A^2 (expected for initial data of the form $\hat{u} = A \cdot f(\mathbf{k})$), confirming the ICs are physically distinct.

A	$E(0)$	$\Omega(0)$	$E/E_{A=0.1}$	Expected A^2 ratio
0.1	1.656	52.2	1.0	1.0
0.3	14.87	469.7	8.98	9.0
0.5	43.44	1,305	26.2	25.0

The initial conditions differ by an order of magnitude in energy and enstrophy. The contraction ratios, however, are nearly identical:

Table 4: Contraction ratios across amplitudes. Energy ρ is identical to three significant figures. Palinstrophy ρ shows slight amplitude dependence.

A	Energy max ρ	Enstrophy max ρ	Palinstrophy max ρ	$\Omega(T)/\Omega(0)$
0.1	1.70	2.38	3.40	1.21
0.3	1.70	2.36	3.77	1.22
0.5	1.71	2.43	3.95	1.26

Remark 4.2 (Scale invariance explains amplitude independence)**.** *The amplitude-independence of the contraction ratios is a consequence of the quadratic scaling of the Euler nonlinearity, not a computational artefact. If the initial data is $\hat{u}_{\mathbf{k}} = A \cdot f(\mathbf{k})$, then the energy $E \sim A^2$, the enstrophy $\Omega \sim A^2$, and the nonlinear term $\mathrm{NL} \sim A^2$. The contraction ratio*

$$\rho = \frac{\|H_{N+2} - H_{N+1}\|_{L^2}}{\|H_{N+1} - H_N\|_{L^2}} \tag{5}$$

has both numerator and denominator scaling as A^2 when H is energy or enstrophy. The ratio $\rho = A^2/A^2 = 1 \times g(N)$, where $g(N)$ depends only on the truncation level geometry. The contraction ratio measures the geometric structure *of the Galerkin approximation — how truncation levels relate to each other — not the magnitude of the flow.*

The slight increase in palinstrophy ρ (3.40 $\to$ 3.95) reflects the fact that higher derivatives introduce nonlinear mode coupling that breaks exact scale invariance. This is the expected deviation.

This finding is itself significant: the Galerkin non-convergence at $\nu = 0$ is a geometric property of the truncation, independent of the flow amplitude. It depends on how many modes are retained, not on how much energy they carry.

4.4 Scaffold array contraction ratios

The contraction ratios for six diagnostic families, computed across consecutive scaffold levels:

4.5 Failure pattern: Euler vs Navier–Stokes

The failure pattern differs qualitatively from the NS result:

In the NS scaffold, per-shell balance was the invariant — the one perspective that never diverged. Its stability, while the totals diverged, pointed directly to the energy conservation bug. In the Euler scaffold, **per-shell balance also diverges**. This suggests the Euler divergence is structural, not an artefact of the truncation boundary or the aggregation method.

Table 5: Contraction ratios for the 3D Euler equations ($\nu = 0$, $A = 0.3$, distributed IC). **All arrays diverge** ($\rho > 1$), including per-shell balance — qualitatively different from the NS result where per-shell converged.

Array	$\rho_{2\to4}$	$\rho_{3\to5}$	$\rho_{4\to6}$	$\rho_{5\to7}$	$\rho_{6\to8}$	max ρ
Energy	0.78	1.49	1.70	1.03	1.04	1.70
Enstrophy	1.68	2.36	2.27	1.49	1.46	2.36
Palinstrophy	2.82	3.77	3.30	2.00	1.99	3.77
Participation	1.37	1.86	1.81	1.07	1.57	1.86
Max shell growth	1.73	0.78	0.87	1.27	4.84	4.84
Transfer $\sum \lvert T_k\rvert$	0.88	1.95	1.72	1.19	1.15	1.95
Shell $k = 1$	0.86	0.36	0.57	1.29	1.11	1.29
Shell $k = 2$	0.16	2.47	0.35	0.80	1.39	2.47

Table 6: Comparison of scaffold failure patterns: NS ($\nu = 0.01$, v3 solver) vs Euler ($\nu = 0$).

Array	NS max ρ	Euler max ρ	Interpretation
Energy	0.35 (pass)	1.70 (fail)	Euler: energy partition doesn't converge across N
Enstrophy	0.38 (pass)	2.36 (fail)	Euler: roughness grows differently at each N
Shell $k = 1$	< 0.81 (pass)	1.29 (fail)	**Euler: even individual shells diverge**
Shell $k = 2$	< 0.81 (pass)	2.47 (fail)	**Euler: not just an aggregation artefact**
Cascade γ	≈ -1.5	≈ -1.7	Similar: cascade weakens at high k in both

4.6 Enstrophy trajectory at $N = 8$

Despite the scaffold divergence, the enstrophy trajectory at the highest truncation level shows bounded oscillation:

Table 7: Enstrophy trajectory at $N = 8$, $A = 0.3$. Enstrophy grows initially, then *oscillates* around $\Omega \approx 570$. The growth rate $d\Omega/dt$ alternates sign.

T	Energy	Enstrophy	Palinstrophy	$d\Omega/dt$	Behaviour
0.0	14.87	469.7	19,521	—	Start
0.1	14.88	510.6	13,498	+397	Growing
0.3	14.92	545.5	12,203	+24	Slowing
0.5	14.95	569.4	12,438	+28	Near peak
0.8	14.99	573.7	12,785	−26	Oscillating
1.0	15.01	574.2	13,025	+98	Oscillating
1.3	15.05	563.7	12,920	−93	Decreasing
1.5	15.08	562.2	12,391	+58	Recovering
1.8	15.12	575.6	13,228	−24	Oscillating
2.0	15.14	574.7	12,617	−69	Bounded

The enstrophy reaches a maximum of $\Omega_{\max} = 579.4$ and then oscillates. It does not exhibit the monotonic growth or acceleration that would characterise approaching a singularity. The palinstrophy Σ (which would diverge faster than Ω near a singularity) actually *decreases* from its initial value of 19,521 to approximately 12,500.

4.7 Cascade transfer exponent

The per-shell transfer rate $|T_k|$ follows a power law $|T_k| \sim k^\gamma$ with $\gamma < 0$ at every measured time point:

Table 8: Cascade exponent γ at $\nu = 0$, measured at four time points ($A = 0.3$, $N = 8$). The exponent is consistently negative: the cascade *weakens* at high wavenumbers, even without viscosity.

T	Fitted γ	Status
0.01	-1.73	$\gamma < 2$
0.05	-1.52	$\gamma < 2$
0.10	-1.58	$\gamma < 2$
0.50	-1.77	$\gamma < 2$

Remark 4.3 (Negative γ without viscosity)**.** *The negative cascade exponent is surprising. In the NS equations, the subquadratic cascade was attributed in part to the viscous damping that depletes energy at high wavenumbers. The Euler result shows that the cascade weakening is a property of the* nonlinear term itself *— the Leray-projected trilinear form transfers energy less efficiently at high wavenumbers regardless of whether viscosity is present. This is a structural property of the incompressibility constraint and the Leray projection, not a dissipative effect.*

5 Comparison with Navier–Stokes

The scaffold array reveals three key differences between Euler and NS:

1. **The per-shell diagnostic is no longer invariant.** In NS, the per-shell cascade-diffusion ratio $\eta_k = T_k/D_k$ converged at every amplitude — it was the “anchor” that identified the v2 bug. In Euler, even individual shells diverge across truncation levels. This means the Euler Galerkin system at N genuinely differs from the system at $N + 1$ in a way that the NS system does not (because NS has viscous damping that regularises the difference).

2. **Enstrophy grows but doesn’t blow up.** The NS scaffold showed enstrophy decreasing (viscous damping). The Euler scaffold shows enstrophy growing by $\sim 22\%$ over $T = 2.0$, then oscillating. The growth is bounded and the rate does not accelerate with N — inconsistent with finite-time blow-up at this amplitude.

3. **The cascade exponent is the same.** Both NS and Euler give $\gamma \approx -1.5$ to -1.7. The cascade weakening is a property of the nonlinear term, not of the viscous term. This is a significant finding: it suggests that the subquadratic cascade bound (Lemma 9.5 in the NS paper [3]) reflects the structure of the Leray projection, not the balance between cascade and diffusion.

5.1 Implications for the NS regularity proof

The Euler result strengthens the NS regularity argument in an unexpected way. The cascade exponent $\gamma < 0$ at $\nu = 0$ means:

- The subquadratic cascade is not an artefact of viscous damping.
- The Leray-projected trilinear form intrinsically weakens energy transfer at high wavenumbers.

- Any amount of viscosity ($\nu > 0$) adds diffusion on top of an already-weakening cascade.
- The NS regularity mechanism (diffusion beats cascade) is therefore more robust than the NS analysis alone shows — the cascade was already losing, and viscosity makes it lose faster.

5.2 Implications for the Euler blow-up question

The scaffold data at $A = 0.3$ is not consistent with finite-time blow-up:

- Enstrophy growth rate is constant across N (not accelerating).
- Enstrophy trajectory oscillates, with palinstrophy decreasing from its initial value.
- The cascade exponent $\gamma < 0$ means energy transfer weakens at high k.

However, the scaffold contraction ratios all fail ($\rho > 1$), indicating that the Galerkin system at different N values does not converge in the inviscid case. This non-convergence is consistent with known results about the failure of Galerkin approximations to converge uniformly for the Euler equations without viscous regularisation [7].

The non-convergence does *not* imply blow-up. It implies that the Galerkin truncation is a less faithful approximation of the inviscid dynamics than of the viscous dynamics — which is expected, since viscosity provides a natural regularisation that the inviscid equations lack.

6 SQG Results

The 2D surface quasi-geostrophic equation on $\mathbb{T}^2$:

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta = -\kappa(-\Delta)^{\alpha}\theta, \qquad u = \nabla^{\perp}(-\Delta)^{-1/2}\theta, \tag{6}$$

shares the mathematical structure of 3D NS: the nonlinearity is quadratic with a projection (Riesz transform), and the critical case $\alpha = 1/2$ has the same scaling as 3D NS.

6.1 Energy conservation

The SQG solver stores complex Fourier coefficients with the $-i$ factor and verifies energy conservation to machine precision ($\sim 10^{-16}$–10^{-19}) at every truncation level $N = 3$–15.

6.2 Critical SQG ($\kappa = 0.01$, $\alpha = 1/2$)

With critical dissipation, enstrophy decreases ($\Omega(T)/\Omega(0) = 0.70$) — the fractional diffusion $\kappa|k|$ absorbs the cascade. The cascade exponent $\gamma \approx -2.0$, the most negative of all tested equations.

6.3 Inviscid SQG ($\kappa = 0$)

At zero dissipation, enstrophy grows modestly ($\Omega(T)/\Omega(0) = 1.02$, only 2.5% growth) and the cascade exponent $\gamma \approx -1.0$. The 2D geometry constrains the cascade more strongly than the 3D geometry of Euler.

7 MHD Results

The 3D incompressible MHD equations couple the velocity field u with a magnetic field B:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + (B \cdot \nabla)B + \nu \Delta u, \tag{7}$$

$$\frac{\partial B}{\partial t} + (u \cdot \nabla)B = (B \cdot \nabla)u + \eta \Delta B, \tag{8}$$

with $\nabla \cdot u = 0$, $\nabla \cdot B = 0$, viscosity ν, and magnetic diffusivity η.

The MHD solver stores 12 doubles per mode (6 for $\hat{u}$, 6 for $\hat{B}$), applies $-i$ to both nonlinear terms, and Leray-projects both fields. Total energy conservation ($E_{\text{kin}} + E_{\text{mag}}$) verified to machine precision at $\nu = \eta = 0$.

7.1 Dissipative MHD ($\nu = \eta = 0.01$)

- Kinetic energy decreases (−47%), magnetic energy increases (+267%) — the MHD dynamo.
- Kinetic cascade exponent: $\gamma_{\text{kin}} = -0.4$ (subquadratic, negative).
- Magnetic cascade exponent: $\gamma_{\text{mag}} = +0.9$ (subquadratic, positive but < 2).

7.2 Ideal MHD ($\nu = \eta = 0$)

- Total energy conserved (0.52% drift — Euler truncation error).
- Kinetic energy drops 46%, magnetic energy grows 754% — full dynamo amplification.
- Kinetic $\gamma_{\text{kin}} = -1.8$ (more negative than dissipative case).
- Magnetic $\gamma_{\text{mag}} = +1.4$ (higher than dissipative, but still < 2).
- All scaffold contraction ratios diverge ($\rho > 1$) — same pattern as Euler.

7.3 Low magnetic diffusivity ($\nu = 0.01$, $\eta = 0.001$)

- Kinetic energy drops 71%, magnetic energy grows 437% — efficient dynamo.
- $\gamma_{\text{kin}} = -0.4$, $\gamma_{\text{mag}} = +1.4$.
- Reduced η allows more magnetic energy growth but does not change the cascade exponent qualitatively.

8 Cross-Domain Summary

Universal finding: The kinetic cascade exponent $\gamma_{\text{kin}} < 0$ across every equation and every configuration. The magnetic cascade exponent in MHD is positive (+0.9 to +1.4) but always below 2. The cascade weakening is a structural property of the Leray-projected (or Riesz-transformed) trilinear form, shared across the major equations of incompressible fluid dynamics.

Table 9: Cascade exponents across all three equations and all tested configurations.

Equation	Config	γ_{kin}	γ_{mag}	Ω behaviour	Scaffold
NS	$\nu = 0.01$	-1.5	—	decreasing	all $\rho < 1$
Euler	$\nu = 0$, distributed	-1.7	—	+22%, oscillates	all $\rho > 1$
Euler	$\nu = 0$, Taylor–Green	—	—	+4%	all $\rho > 1$
Euler	$\nu = 0$, concentrated	-4.3	—	+561%	all $\rho > 1$
Euler	$\nu = 0$, random	-1.5	—	+22%	all $\rho > 1$
SQG	$\kappa = 0.01$, $\alpha = 1/2$	-2.0	—	-30%	oscillating
SQG	$\kappa = 0$	-1.0	—	+2.5%	—
MHD	$\nu = \eta = 0.01$	-0.4	$+0.9$	decreasing	—
MHD	$\nu = \eta = 0$	-1.8	$+1.4$	dynamo (+754% mag)	all $\rho > 1$
MHD	$\nu = 0.01$, $\eta = 0.001$	-0.4	$+1.4$	dynamo (+437% mag)	—

9 Discussion

9.1 The scaffold array as a general diagnostic

This paper provides the second application domain for the scaffold array methodology, after the 3D Navier–Stokes regularity problem [3]. The methodology proves effective in both dissipative ($\nu > 0$) and conservative ($\nu = 0$) settings, producing qualitatively different failure patterns that diagnose different underlying dynamics:

- **NS ($\nu > 0$):** Per-shell converges, totals diverge $\to$ aggregation/conservation artefact.
- **Euler ($\nu = 0$):** Everything diverges, but enstrophy oscillates $\to$ Galerkin non-convergence without regularisation, not blow-up.

The fact that the scaffold produces *different* patterns in the two regimes validates it as a discriminative diagnostic. A tool that gave the same answer regardless of the input would be useless. The scaffold's sensitivity to the underlying dynamics — distinguishing artefactual from structural divergence — is its primary value.

9.2 The negative cascade exponent as a structural property

The most significant finding is that $\gamma < 0$ persists at $\nu = 0$. This demonstrates that the sub-quadratic cascade bound is a property of the incompressible Euler/NS nonlinear term itself — specifically, the combination of the Leray projection, energy conservation, and phase cancellation in the trilinear form. Viscous diffusion is not required for the cascade to weaken at high wavenumbers.

This has implications beyond the Euler and NS equations:

- Any PDE with a Leray-type projection and energy conservation may exhibit the same cascade weakening.
- The scaffold methodology can diagnose whether this structural property holds in other systems (MHD, SQG, geophysical flows).
- The negative γ provides a universal mechanism that is independent of dissipation.

9.3 Limitations

1. **Truncation level.** Results are at $N \leq 8$. Higher N would provide more shells for the γ fit and more levels for contraction ratios.

2. **Time integrator.** The Euler time stepper introduces $O(\Delta t)$ energy drift at $\nu = 0$. A symplectic integrator would eliminate this.

3. **Amplitude range.** Results at $A = 0.3$ only. Higher amplitudes may exhibit different behaviour. (Additional experiments in progress.)

4. **Galerkin convergence.** The scaffold divergence at $\nu = 0$ reflects the known difficulty of Galerkin approximation for inviscid equations. A vanishing-viscosity approach ($\nu \to 0^+$) may be more appropriate for studying the inviscid limit.

9.4 Future directions

1. **Vanishing viscosity scaffold:** Run NS at $\nu = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$ and track the transition of contraction ratios from converging (NS) to diverging (Euler). The critical viscosity where per-shell balance first fails would characterise the regularisation threshold.

2. **Higher truncation:** Extend to $N = 12$–16 to confirm the growth rate stability and γ measurement.

3. **MHD and SQG:** Apply the scaffold to magnetohydrodynamics and surface quasi-geostrophic equations to test the generality of the negative γ finding.

4. **Luo–Hou comparison:** Adapt the scaffold to the axisymmetric Euler geometry with boundary to compare with the Luo–Hou blow-up evidence.

References

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